Random walks of partons in $SU(N_c)$ and classical representations of color charges in QCD at small x

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Abstract

The effective action for wee partons in large nuclei includes a sum over static color sources distributed in a wide range of representations of the $SU(N_c)$ color group. The problem can be formulated as a random walk of partons in the $N_c - 1$ dimensional space spanned by the Casimirs of $SU(N_c)$. For a large number of sources, $k \gg 1$, we show explicitly that the most likely representation is a classical representation of order $\mathrm{O}(\sqrt{k})$. The quantum sum over representations is well approximated by a path integral over classical sources with an exponential weight whose argument is the quadratic Casimir operator of the group. The contributions of the higher $N_c - 2$ Casimir operators are suppressed by powers of k. Other applications of the techniques developed here are discussed briefly.

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1   Introduction

In QCD, there are a number of physical situations where higher dimensional representations of the gauge group are relevant. In particular, in perturbative QCD, because of the ubiquitous nature of bremsstrahlung processes, one often encounters situations where partons at one energy scale interact simultaneously with several partons at a different energy scale [1]. The former may therefore couple to color charges in a wide range of representations of the color group. If one can argue that higher dimensional representations are the most likely representations, it may be possible, by the correspondence principle, to treat these as classical representations. Classical color charges are considerably simpler to treat than their quantum counterparts.

In this paper, we will address the following formal questions which are relevant to addressing this issue quantitatively. Given k non-interacting quarks in the fundamental $SU(N_c)$ representation, what is the distribution of degenerate irreducible representations that one generates? What is the most likely representation? Is this representation a classical representation? It is well known that the $N_c \to \infty$ limit of QCD is a classical theory even for $k = 1$ [2]. What happens for finite N_c but $k \gg 1$ [1]? We will demonstrate in the following that, for an $SU(N_c)$ gauge theory, the problem can be formulated as a random walk problem in the space spanned by the $N_c - 1$ Casimir operators of the group.

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[1]The former corresponds to the large N limit where the invariance group of the theory grows with N. Several such large N cases have been discussed in Ref. [2]. We are instead interested in a situation where the underlying symmetry group is unchanged but where quantum operators in higher dimensional representations appear. Examples of these are the large spin limits of quantum spin models [3].

Our interest in the questions posed here arise in the context of a particular model of small x physics in QCD, the McLerran-Venugopalan (MV) model for small x parton distributions in large nuclei [4]. The MV model is a simple model to understand the physics of saturation [5] at small x in QCD. To highlight the physical relevance of the formal mathematics, we will therefore address solutions to these general questions in the specific context of the MV model. However, the solution to the mathematical problem posed may be of more general interest and will hopefully prove useful in developing novel treatments for a variety of physical situations in QCD. These include problems in jet physics [6], finite temperature transport theory [7] and even percolation and string based models of multi-particle production [8, 9].

This paper is organized as follows. In the following section, we will briefly review the McLerran-Venugopalan model. We will focus in particular on those aspects of the model related to the random walk of partons in the space spanned by the Casimirs of their color group and of their subsequent treatment as classical color charge distributions. In section 3, we consider the case of the MV-model in an $SU(2)$ gauge theory. The problem here is especially simple since it can be mapped into a one dimensional random walk problem of k spins of spin-1/2. The peak of the distribution is at a representation whose dimension is of order O($\sqrt{k}$). The distribution of representations about this peak value is well approximated as an exponential in the quadratic Casimir operator weighted by k, as indeed assumed in the MV-model.

The "real world" case of an $SU(3)$ gauge theory is discussed in section 4. In this case, one can think of the random walk as occurring in the space

spanned by the third component of the isotopic spin I_3 and the hypercharge Y. Determining the distribution of higher dimensional representations generated by k quarks when the random walk is in more than one dimension ($SU(N_c)$ for $N_c \geq 3$) is non-trivial and is the primary subject of this paper. We will show explicitly, that for a random walk of k quarks in the fundamental $SU(3)$ representation, the distribution of degenerate representations is, as in the $SU(2)$ case, also peaked at a representation of order $O(\sqrt{k})$. The distribution of representations about the most likely representation, however, cannot simply be expressed in terms of an exponential of the quadratic Casimir operator alone; the argument of the exponential has an additional term proportional to the cubic Casimir operator weighted by k^2. The cubic term thus acts as a perturbation of the quadratic term. The relative magnitude of the former to the latter is of order $O(1/\sqrt{k})$ with a coefficient that is computed exactly. Thus for very large nuclei $A \rightarrow \infty$, the contribution from this term can be neglected [2]. We also consider the representations generated by random walks of gluons and of quark-anti-quark pairs. Interestingly, in both of these cases, the distribution of representations is given by an exponential in the quadratic Casimir alone. This is true because the quadratic Casimir, unlike the cubic Casimir, is symmetric in the upper and lower $SU(3)$ tensor indices (denoted by m and n in the paper). Gluons and quark-anti-quark representations, unlike the quark representations, favor representations that are symmetric in m and n.

The generalization of these results to $SU(N_c)$ for any general N_c is straightforward and is briefly discussed in section 5. The essential features of

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[2]The statements here are valid for values of x where small x quantum evolution is not too important. Quantum evolution effects will be discussed briefly later in this section and in the final section.

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the $SU(3)$ result are preserved: the distribution of representations about the most likely representation is determined primarily by the quadratic Casimir. The other $N_c - 2$ Casimirs add small (and in principle quantifiable) perturbative corrections–all of which vanish for $k \to \infty$.

In section 6, we will summarize our results and discuss their ramifications. A brief discussion of classical limits of quantum systems in general, and quantum spin systems in particular, is presented in appendix A. The other appendices contain technical details of results presented in the body of the paper.

2 Classical color charges in the McLerran-Venugopalan model

The McLerran-Venugopalan model is a classical effective field theory for wee parton distributions in large nuclei [4]. The model is formulated in the light cone gauge $A^+ = 0$, and in the infinite momentum frame, where the momentum of the nucleus, $P^+ \to \infty$. In this case, the physics of time dilation ensures that the time scales for partons carrying a higher fraction x of the nuclear momentum (the "valence" partons with $x \sim 1$) to interact with one another is much larger than the characteristic time scale for their interactions with softer "wee" partons (with $x \ll 1$). The softer partons, in turn, couple to a large number of valence partonic sources, which appear static on the wee parton time scales. In particular, wee partons with coherence lengths ($l_{\text{coh.}} \approx 1/2m_N x$) much greater than the Lorentz contracted nuclear widths (of order $2R/\gamma$), or equivalently with $x \ll A^{-1/3}$, couple coherently to $\sim A^{1/3}$ valence partons along the longitudinal length of the nucleus in the

infinite momentum frame. Here m_N is the nucleon mass and $\gamma \sim P^+/m_N$ is the Lorentz factor in the infinite momentum frame.

The separation of scales in x between wee and valence partons is experimentally well established in parton distribution function measurements [10]. The insensitivity of the valence distributions to the sea is also experimentally well established-the well known phenomenon of "limiting fragmentation" is a direct consequence [11].

Further, on the time scales relevant to wee parton dynamics, it is reasonable to assume that the valence parton sources are random light cone sources. This is plausible because firstly, most of the hard partons are confined in different nucleons (and hence do not interact with one another); secondly, due to time dilation, even hard partons in the same nucleon are non-interacting (and therefore independent sources) over the short time scales relevant to the wee partons.

How many of these random sources the wee partons actually couple to depends on the typical transverse momentum of the wee parton [3]. A wee parton with momentum $p_\perp$ resolves an area in the transverse plane $(\Delta x_\perp)^2 \sim 1/p_\perp^2$. The number of valence partons it interacts simultaneously with is then

$$k \equiv k_{(\Delta x_\perp)^2} = \frac{N_{\text{valence}}}{\pi R^2} (\Delta x_\perp)^2, \tag{1}$$

which indeed is proportional to $A^{1/3}$ since $N_{\text{valence}} = 3 \cdot A$ in QCD. This counting of color charges is only valid as long as $p_\perp > \Lambda_{\text{QCD}}$ since, on the scale of the nucleon size , $p_\perp \sim \Lambda_{\text{QCD}} \sim 200\,\text{GeV}$, confinement ensures that the wee partons see no net color charge. The momentum scale for color neutrality may

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[3]The wee parton is soft only in its longitudinal momentum-its transverse momentum may be large.

be even larger than the confinement scale due to screening effects analogous to Debye screening [12, 13]. These effects, while very interesting, are beyond the scope of this paper.

It was argued in the MV model that when $k \sim A^{1/3} \gg 1$, the most likely color representation that the wee partons couple to is a higher dimensional representation. We will demonstrate later that this representation is one of order $O(\sqrt{k})$. Thus, as discussed further in appendix A, for large enough k, the color charge distribution can be treated as a classical color charge distribution. It is further argued in the MV-model, that the distribution of classical representations is Gaussian, with a variance proportional to k (or, equivalently, to $A^{1/3}$).

With these stated assumptions, the classical effective Lagrangian for the MV-model, formulated in the infinite momentum frame ($P^+ \rightarrow \infty$) and light cone gauge ($A^+ = 0$) has the form

$$\mathcal{L} = \int d^4x \left[\frac{1}{4} F^a\, F^a - J \cdot A\right] + i \int d^2 x_t \frac{\rho^a \rho^a}{2\mu_A^2}\,.$$

$$(2)$$

The first term is the usual QCD Field Strength tensor squared (Lorentz indices are suppressed for simplicity here), which describes the dynamics of the wee partons. The second term denotes the coupling of these wee partons to the hard valence parton sources. The valence parton current has the form

$$J^{\mu,a} = \rho^a(x_t)\delta(x^-)\delta^{\mu+}\,,$$

$$(3)$$

where ρ is the *classical* color charge per unit transverse area of the valence parton sources [4]. The likelihood of configurations of differing ρ's is given by

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[4]As written, this term in the action is not gauge invariant. Gauge invariant expressions are given in Refs. [14, 16]. The lowest order in g term in the expansion of these expressions gives the form shown here.

the final term in Eq. (2) – the weight μ_A^2 of the Gaussian is the average color charge squared per unit area per color degree of freedom. The correlator of color charge densities is then [5]

$$\left\langle \rho^a(x_\perp)\rho^b(y_\perp) \right\rangle = \mu_A^2 \, \delta^{ab} \, \delta^{(2)}(x_\perp - y_\perp) \, . \tag{4}$$

The average color charge squared per unit area per color degree of freedom, μ_A^2 is simply determined in the MV-model. The color charge squared in a tube of transverse area $(\Delta x_\perp)^2$ is the color charge squared per unit quark $(g^2 C_F)$ times the number of quarks in the tube: $(\Delta x_\perp)^2 \cdot A \, N_c/\pi R^2$. When this is normalized as in Eq. (4), per unit transverse area, per color degree of freedom, one obtains

$$\mu_A^2 = \frac{g^2 A}{2\pi R^2} \, , \tag{5}$$

which is of order $A^{1/3}$. For a very large nucleus, $\mu_A^2 \gg \Lambda_{\rm QCD}^2$, and since it is the only scale in the problem, the coupling constant $\alpha_s \equiv \alpha_s(\mu_A^2) \ll 1$. Since the coupling is weak, one can compute parton distributions for a large nucleus. The classical field equations can be solved and it was shown explicitly that the number distribution is of order $1/\alpha_S$ for $p_\perp^2 < g^2 \mu_A^2$ [18].

The Gaussian functional weights and the classical field equations for a large nucleus are also recovered in a particular model of large nuclei where the nucleons are modeled as color singlet quark anti-quark pairs [19]. In Refs. [18] and [19], it was recognized that it was essential to smear the δ-function sources in Eq. (3) in the x^- direction to obtain regular classical solutions. Quantum corrections to the MV-model are large [20] but they can

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[5]There are several conventions for the color charge densities in the literature. For a discussion, see Ref. [17].

be absorbed, via a Wilsonian renormalization group procedure, at each step in x into new sources and fields, while preserving the essential structure of the classical effective Lagrangian [14, 21]. Thus this simple MV-model can be reformulated as a more sophisticated effective field theory, the "Color Glass Condensate" (CGC), which is applicable to both hadrons and nuclei. In the CGC, the weight functional can be represented more generally as

$$\exp\left(-W[\rho]\right), \tag{6}$$

where $W[\rho]$ is a gauge invariant functional that satisfies the non-linear renormalization group (RG) equation in x. For a recent review of the MV model and the CGC, see Ref. [22].

In the rest of the paper, we will restrict ourselves to the MV model and seek to establish more rigorously the following, namely,

- the most likely representation the wee partons couple to is a higher dimensional representation which can be represented in terms of a classical color charge density ρ,

- the sum over color charges in the path integral can be expressed as a path integral over classical color charges, with a weight that is well approximated by a Gaussian for a large number k of color charges. This is shown for quark sources as well as for gluon and quark–anti-quark sources for $N_c \geq 3$.

- Finally, we will discuss, for $N_c \geq 3$, possible sub-leading corrections to the Gaussian term and their relative dependence on k.

3 Random walks and classical color charge representations in two color QCD at small x

We now consider the case of the MV-model in $SU(2)$ QCD. As discussed previously, (see Eq. (1)) the wee partons couple to a large number of uncorrelated quarks-in this case, quarks in the fundamental $SU(2)$ representation. We wish to discuss here the distribution of representations generated by adding k-quarks.

3.1 Random walk of many spin-1/2 quarks

The problem of adding k random color charges in $N_c = 2$ QCD is exactly equivalent to the problem of adding k spins of spin-1/2–they both correspond to an internal symmetry group whose generators are elements of the $SU(2)$ algebra. Therefore, let's start with a spin 0 singlet state. Multiplying with a spin 1/2 state results in a 1/2 state, or

$$\mathbf{2} \times \mathbf{1} = \mathbf{2} \tag{7}$$

where $\mathbf{1}$ is the singlet state and $\mathbf{2}$ denotes the spin 1/2 state. In general we denote a spin l state by the degrees of freedom associated with the state, namely, $\boldsymbol{s} = 2l + 1$.

Continuing, multiplying another spin 1/2 states results in

$$\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{3} \tag{8}$$

If we have 3 spin 1/2's, we get

$$\mathbf{2} \times \mathbf{2} \times \mathbf{2} = \mathbf{2} \times (\mathbf{1} + \mathbf{3}) = \mathbf{2} + \mathbf{2} + \mathbf{4} \,, \tag{9}$$

and so on.

Let $v_s^{(k)}$ denote the multiplicity of the representation $\boldsymbol{s}$ when k fundamental representations are multiplied. In general, if one adds one more spin $1/2$ particle to an $\boldsymbol{s}$ state, the result is a mixture of the states $\boldsymbol{s} - \boldsymbol{1}$ and $\boldsymbol{s} + \boldsymbol{1}$:

$$2 \times \boldsymbol{s} = (\boldsymbol{s} - \boldsymbol{1}) + (\boldsymbol{s} + \boldsymbol{1}) \tag{10}$$

Therefore, the multiplicity of $\boldsymbol{s}$ state in the k-th iteration is given by

$$v_s^{(k)} = v_{s-1}^{(k-1)} + v_{s+1}^{(k-1)} \tag{11}$$

We wish to find the solution of this recursion relation with the initial condition

$$v_0^{(0)} = 1 \quad \text{otherwise} \quad v_s^{(0)} = 0 \tag{12}$$

We begin by noting that the binomial coefficients satisfy a similar (but not identical) recursion relation,

$$\binom{k}{s} = \binom{k-1}{s-1} + \binom{k-1}{s}. \tag{13}$$

Therefore, we can try and find a solution to Eq. (11) in terms of binomial coefficients.

Define

$$G_{k:\,s} = \binom{k}{(k+s)/2}. \tag{14}$$

which is non-zero only if $(k+s)$ is even. Consider now the combination

$$
\begin{aligned}
g(k:s) &= G_{k:\,s-1} - G_{k:\,s+1} \\
&= \binom{k}{(k+s-1)/2} - \binom{k}{(k+s+1)/2}.
\end{aligned}
\tag{15}
$$

Using the binomial coefficient relation (13), one can easily show that this combination satisfies the recursion relation

$$g(k:s) = g(k-1:s-1) + g(k-1:s+1) \tag{16}$$

Furthermore, for $k = 0$

$$g(0:1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \tag{17}$$

and all the other $g(0:s)$'s are zero.

In this way, we have found a solution to the recursion relation (11) satisfying the right boundary condition, Eq. (12), namely,

$$v_s^{(k)} = g(k:s) = G_{k:s-1} - G_{k:s+1} \tag{18}$$

Since we know the G's explicitly in terms of binomial coefficients (Eq. (14)), we can determine $v_s^{(k)}$ explicitly. Since we are interested in the limit $k \gg s \gg 1$, we can use the well known Stirling formula to approximate

$$G_{k:s} = \frac{k!}{((k+s)/2)!\,((k-s)/2)!} , \tag{19}$$

as

$$\ln G_{k:s} \approx k \ln 2 - s^2/2k - (1/2) \ln k - (1/2) \ln(2\pi) , \tag{20}$$

or equivalently,

$$G_{k:s} \approx \frac{2^{k-1/2}}{\sqrt{k\pi}} e^{-s^2/2k} . \tag{21}$$

Thus the solution of the recursion relation for $k \gg s \gg 1$ is

$$\begin{aligned} v_s^{(k)} &= G_{k:s-1} - G_{k:s+1} \\ &\approx -2\partial_s G_{k:s} \\ &\approx \frac{2^{k+1/2}}{k\sqrt{k\pi}} s\, e^{-s^2/2k} \end{aligned} \tag{22}$$

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This formula represents the distribution of representations with spin $l = (s-1)/2$ when there are k random spin $1/2$ particles in the system.

To find the *probability* of the spin state l, we need to take into account the degeneracy of the spin state $s = 2l + 1$. Since we have k random spin $1/2$ particles, summing the degeneracy times the multiplicity $(s \, v_s^{(k)})$ over all possible representations must be equal to the 2^k degrees of freedom. Even with our large k approximation, we find that this is indeed the case:

$$\int_0^\infty ds \, s \, v_s^{(k)} = 2^k \ . \tag{23}$$

Defining the "classical" spin vector [6] $\mathbf{l}$ which is related to s by the relation

$$s^2 = 4 \, \mathbf{l}^2 \ , \tag{24}$$

or equivalently $\mathbf{l}^2 = (l + 1/2)^2$, we can exploit the lack of angular dependence in the argument to re-write Eq. (23) as

$$1 = \left(\frac{2}{k\pi}\right)^{3/2} \int d^3 l \exp\left(-2 \, \mathbf{l}^2/k\right) \ . \tag{25}$$

Thus one can define

$$\mathcal{P}_k(\mathbf{l}) = \left(\frac{2}{k\pi}\right)^{3/2} e^{-2\mathbf{l}^2/k} \ , \tag{26}$$

to be the probability density for the system of k spin-$1/2$ particles to have the total spin $\mathbf{l}$. The above formula clearly resembles the classical Maxwell-Boltzmann distribution of particles in a heat bath.

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[6]We will see *a posteriori* that $\mathbf{l}^2$ is of order k. For large k, we argue in Appendix A that such representations are classical representations.

Our result can be easily expressed in terms of the $SU(2)$ Casimir, which is

$$D_2 = l(l+1) \approx \mathbf{l}^2 \,.\tag{27}$$

The most probable value of the Casimir is then given by

$$\bar{D}_2 = \frac{k}{4}\tag{28}$$

or $l = \sqrt{k}/2 + O(1)$.

One can repeat the same analysis for the case of many spin-1 particles. In this case, one has a somewhat more complex analysis (involving trinomial coefficients) but the final result for the multiplicity distribution is identical to the spin-1/2 case-up to overall constants.

What does this result for $v_s^{(k)}$ imply for the derivation of the effective action in the MV model? In order to understand this, we have to step back a little and discuss, with some greater detail, the derivation of the MV-effective action [4].

3.2 From random walks to path integrals

The path integral describing the ground state $|O\rangle$ of a large nucleus can be written as

$$\mathcal{Z} = \langle O|\, e^{i\,x^+\, P_{\mathrm{QCD}}^-}\, |O\rangle = \lim_{x^+ \to i\infty} \sum_{N,Q} \langle N,Q|\, e^{i\,x^+\, P_{\mathrm{QCD}}^-}\, |N,Q\rangle \,,\tag{29}$$

The sums over N and Q here respectively represent the sum over all possible states in the path integral and a sum over the color quantum numbers of

these states [7]. In the lattice representation of the path integral, a state $|N, Q\rangle$ would correspond to a 4-dimensional box on the lattice containing a net color charge Q. In the MV-model, one is interested in a coarse grained theory, where the size of the box is chosen (see the first paper in Ref. [4]) such that it contains a large number of quark color charges, namely, $k \gg 1$. From Eq. (1), this implies that the coarse grained state $|N, Q\rangle$ contains only modes with $\Lambda_{\text{QCD}} \ll p_\perp^2 \ll \mu_A^2$ (see Eq. (5)) and the corresponding color charge Q is the charge constructed from all the $k_{(\Delta x_\perp)^2}$ quarks contained in a box with transverse size $1/\mu_A^2 \ll (\Delta x_\perp)^2 \ll 1/\Lambda_{\text{QCD}}^2$.

In general, one has to perform the quantum mechanical sum over the color charges to construct $|N, Q\rangle$, which is a difficult problem indeed. It is at this point one sees the relevance of the problem of the distribution of representations that we have just solved (for the $N_c = 2$ case). In the MV model, the color charges are random, and k charges can be distributed in a wide range of distributions. Since the most likely representation is $\bar{s} = \sqrt{k}/2 \gg 1$ for $k \gg 1$, the color charge corresponding to this representation, as argued in appendix A, is a classical representation.

Thus in the $SU(2)$ case, the sum over all spin states in the path integral can be replaced by the integral [8]

$$\sum_l v_l^{(k)} \sum_{m=-l}^l |l, m\rangle \langle l, m| \rightarrow \int d^3 l \, e^{-2\mathbf{l}^2/k} \, , \tag{30}$$

where we have made use of Eqs. (22) and (26). For a transverse area of size $(\Delta x_\perp)^2$ containing k spins, then one can introduce classical color charge

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[7]The distinction between the quantum numbers represented by N and the color degrees of freedom represented by Q is of course artificial. We make this distinction because it will prove useful in constructing the effective theory.

[8]This assumes that the $s = 2l + 1$ states in a representation have the same P^+.

(spin) density ρ^a which is defined by the relation,

$$l_a = \frac{1}{g} \int_{(\Delta x_\perp)^2} d^2 x_\perp \, \rho_a(x_\perp) \approx \frac{(\Delta x_\perp)^2}{g} \rho_a(x_\perp) \,. \tag{31}$$

We can now re-express the Gaussian weight in Eq. (30) in terms of the classical color charge density:

$$\begin{aligned} 2\frac{\mathbf{l}^2}{k} &= N_c \frac{\mathbf{l}^2}{k} \\ &= N_c \frac{(\Delta x_\perp)^4}{g^2 k} \rho_a \rho_a \\ &= \frac{\pi R_A^2}{g^2 A} (\Delta x_\perp)^2 \, \rho_a \rho_a \,. \end{aligned} \tag{32}$$

Here we have used Eq. (1) to express k in terms of the number of valence quarks in an area $(\Delta x_\perp)^2$. We also used the fact that the number of valence quarks per baryon is equal to N_c.

The sum in Eq. (30) can therefore be expressed as

$$\int \prod_a d\rho^a \, W[\boldsymbol{\rho}] \equiv \int \prod_a d\rho^a \, \exp\left(-\int_A d^2 x_\perp \frac{\boldsymbol{\rho}(x_\perp) \cdot \boldsymbol{\rho}(x_\perp)}{2\mu_A^2} \right) \tag{33}$$

where

$$\mu_A^2 = \frac{g^2 A}{2\pi R_A^2} \tag{34}$$

is independent of N_c, and is the color charge squared per unit area defined in the MV-model (Eq. (5)). Integrating over all the boxes of area $(\Delta x_\perp)^2$ in the transverse plane, one obtains the path integral for classical color charge distributions in the MV-model [4].

4 Random walks and classical color charge representations for $N_c = 3$ QCD at small x

We shall now extend our discussion in the previous section to the case of three color QCD. The problem is more non-trivial here since one now has a 2-dimensional random walk in the space generated by the two $SU(3)$ Casimirs. Thus it is no longer obvious that the quadratic Casimir alone will contribute to the "Boltzmann" weight in the path integral over color charges. We will show in this section that the approximation that only the quadratic Casimir contributes is an excellent approximation when the number of quarks is large. We will first consider the case where the sources of color charge are $SU(3)$ quarks only before considering the representations generated by random gluons and quark-anti-quark states respectively.

4.1 Many $SU(3)$ quarks

In the $SU(3)$ case, representations are labeled by two integers m and n and written as (m, n). Here m and n are the number of the upper and the tensor indices, respectively. A fundamental quark **3** state in $SU(3)$ is hence $(1, 0)$ while an anti-quark in the $\bar{\mathbf{3}}$ state is $(0, 1)$. As is well known, $SU(3)$ representations can be conveniently represented graphically as Young tableaux, where m denotes the number of boxes in the uppermost row minus those in the middle row, while n is the number of boxes in the middle row minus that of the bottom row.

Recall that we are interested in studying the distribution of representations when one adds k quarks in the fundamental representation. Consider
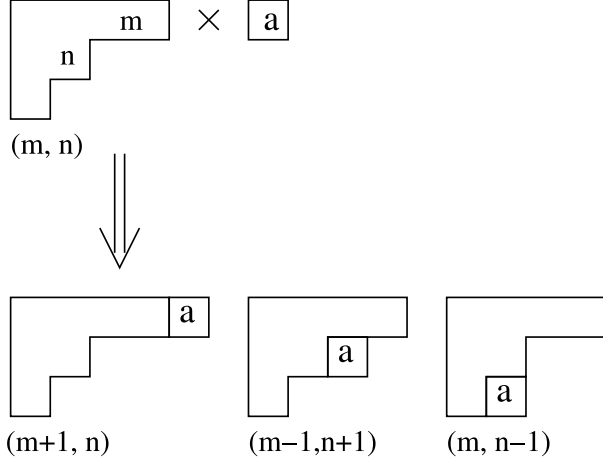
Figure 1: Young tableaux representing the result of multiplying $SU(3)$ quarks in a representation labeled by the tensor indices m & n with a quark in the **3** representation.

what happens when one adds a quark to an (m, n) representation. This is represented by the Young tableaux in Fig. 1 which allows us to deduce that

$$\mathbf{(1,0) \times (m,n) = (m+1,n) + (m-1,n+1) + (m,n-1)} \quad (35)$$

We shall represent the multiplicity of each state by a matrix element $N_{m,n}$ with the indices $0 \le m, n$ representing the same (m, n) of the representations. (Note that these $N_{m,n}$ are the $SU(3)$ analogs of the v_s in the $SU(2)$ case.) The multiplicity of the (m, n) state in the $(k+1)$-th iteration is then given by the recursion relation

$$N_{m,n}^{(k+1)} = N_{m-1,n}^{(k)} + N_{m+1,n-1}^{(k)} + N_{m,n+1}^{(k)} \quad (36)$$

for $m, n \ge 0$ with the understanding that $N_{m,-1}^{(k)} = N_{-1,n}^{(k)} = 0$. The initial condition is

$$N_{0,0}^{(0)} = 1 \quad \text{otherwise} \quad N_{m,n}^{(0)} = 0 \quad (37)$$

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As in the $SU(2)$ case, we now wish to determine $N_{m,n}^{(k)}$ by solving this recursion relation.

We begin by noting that Eq. (36) represents a trifucation process since we are multiplying **3**'s. The basic unit should therefore be the tri-nomial coefficients [9]:

$$C_{k:\, l_1,l_2,l_3} \equiv \frac{k!}{l_1!\, l_2!\, l_3!} \delta_{k-l_1-l_2-l_3} \tag{38}$$

Following the pattern set by Eq. (14) and Eq. (35), we define

$$(m, n) = l_1(1, 0) + l_2(-1, 1) + l_3(0, -1), \tag{39}$$

Re-defining $C_{k;\,l_1,l_2,l_3}$ in terms of m, n as $G_{k;m,n}$, we have,

$$G_{k:\, m,n} = \frac{k!}{\left(\frac{k+2m+n}{3}\right)!\left(\frac{k-m+n}{3}\right)!\left(\frac{k-m-2n}{3}\right)!}\,. \tag{40}$$

We are now in a position to compute the multiplicity of representations. One finds that

$$
\begin{aligned}
N_{m,n}^{(k)} \;=\;\; & G_{k:\, m,n} + G_{k:\, m+3,n} + G_{k:\, m,n+3} \\
& - G_{k:\, m+2,n-1} - G_{k:\, m-1,n+2} - G_{k:\, m+2,n+2}
\end{aligned} \tag{41}
$$

satisfies the recursion relation Eq. (36) and the initial condition Eq. (37). We have verified this analytically and have checked numerically as well that the recursion relations (36) and our solution (41) indeed yield the same values upon iteration.

As performed previously in the $SU(2)$ case, we can use the Stirling formula to write the above expression for G (after some algebra) compactly

---

[9]Trinomial coefficients also arise in the adjoint spin-1 representation of $SU(2)$.

as

$$\ln G_{k:\,m,n} \approx \ln \left(\frac{3^{k+3/2}}{2\pi k} \right) - \frac{(m^2 + mn + n^2)}{k} + \frac{(m-n)(2m+n)(m+2n)}{6k^2} \tag{42}$$

for $k \gg m, n \gg 1$.

Recall that the $SU(3)$ Casimir for a given m, n state is given by [23]

$$D_2^{(m,n)} = \frac{1}{3} \left(m^2 + mn + n^2 \right) + (m+n). \tag{43}$$

Note further that this Casimir is symmetric in m and n. This is unlike the Cubic Casimir which is anti-symmetric under exchange of m and n and can be written as [24]

$$D_3^{(m,n)} = \frac{1}{18} \left(m + 2n + 3 \right) \left(n + 2m + 3 \right) \left(m - n \right) \tag{44}$$

A little algebra should suffice to convince ourselves that G can be simply written in terms of the $SU(3)$ Casimirs as [10]

$$G_{k:\,m,n} \approx \frac{3^{\frac{3}{2}+k}}{2\,k\,\pi} \exp\left(-3D_2^{m,n}/k\right) \left(1 + 3D_3^{m,n}/k^2\right). \tag{45}$$

Clearly, near the peak, one has $m = O(\sqrt{k})$ and $n = O(\sqrt{k})$. Hence the cubic Casimir introduces a correction of size $O(1/\sqrt{k})$ for large k.

In the large k limit, one finds that the multiplicity can be approximated as

$$
\begin{aligned}
N_{m,n}^{(k)} &= G_{k:\,m,n} + G_{k:\,m+3,n} + G_{k:\,m,n+3} \\
&\quad - G_{k:\,m+2,n-1} - G_{k:\,m-1,n+2} - G_{k:\,m+2,n+2} \\
&\approx 2\,\partial_m^3 G_{k:\,m,n} + 2\,\partial_n^3 G_{k:\,m,n} - 3\,\partial_m \partial_n^2 G_{k:\,m,n} - 3\,\partial_n \partial_m^2 G_{k:\,m,n} \\
&\approx \frac{27\,m\,n\,(m+n)\,3^{\frac{3}{2}+k}}{k^3\,2\,k\,\pi} \exp\left(-3\,D_2^{m,n}/k\right),
\end{aligned}
\tag{46}
$$

---

[10]Here we have kept only the leading terms in D_2 and D_3 and expanded the exponential in D_3 to lowest order. Our approximation can be checked to be self-consistent.

where we have dropped the D_3 contribution.

Recall that the dimension of an SU(3) representation is

$$d_{mn} = \frac{(m+1)(n+1)(m+n+2)}{2} \approx \frac{mn(m+n)}{2} . \qquad (47)$$

Therefore, just as in the SU(2) case, the probability to find the system of k quarks in an (m,n) representation should be proportional to $d_{mn} N_{m,n}^{(k)}$. Furthermore, one should recover the total number of degrees of freedom 3^k by summing $d_{mn} N_{m,n}^{(k)}$ over all possible representations.

With the expression obtained so far, one can show that integrating $N_{m,n}^{(k)}$ weighted by d_{mn} over $0 \le m, n \le \infty$ yields,

$$\frac{27 \times 3^{\frac{3}{2}+k}}{4\,k^4\,\pi} \int_0^\infty dmdn\, m^2\, n^2\, (m+n)^2\, \exp\left(-3D_2^{m,n}/k\right) = 3^{k+1} \qquad (48)$$

The discrepancy with the integral above arises from the fact that the actual $N_{m,n}^{(k)}$ is rather asymmetric in m and n [11]. Since we are multiplying k number of $\mathbf{3} = (1,0)$ representations, the m side is more heavily populated than the n side. On the other hand, the approximation given in Eq. (46) is symmetric in m and n which is valid only in the vicinity of the peak.

Nevertheless, our result enables us to normalize the distribution of representations as

$$1 = \frac{27\sqrt{3}}{4\,k^4\,\pi} \int_0^\infty dmdn\, m^2\, n^2\, (m+n)^2\, \exp\left(-3D_2^{m,n}/k\right) \qquad (49)$$

We shall show below that we can re-write Eq. (49), in analogy to Eq. (25), as

$$1 = \left(\frac{N_c}{k\pi}\right)^4 \int d^8Q\, e^{-N_c\,\mathbf{Q}^2/k} \qquad (50)$$

---

[11] Our method yields the correct number of degrees of freedom for gluons and quark-anti-quark pairs which, unlike quarks, are symmetric in m and n (see sections 4.2 and 4.3).

where $\mathbf{Q} = (Q_1, Q_2, \cdots, Q_8)$ is a classical color charge vector defined by $|\mathbf{Q}| = \sqrt{Q^a Q^a} \equiv \sqrt{D_2^{m,n}}$ and $Q_1, \cdots, Q_8$ are its eight components. Once we are able to do this, the rest of the argument, expressing the sum over color charges in large nuclei as a path integral over Gaussian color charges, will follow in smooth analogy to the derivation in section 3.2.

For any $SU(3)$ representation (not necessarily a classical representation), one can write the measure d^8Q as [25, 26, 27]

$$d^8Q = d\phi_1 d\phi_2 d\phi_3 d\pi_1 d\pi_2 d\pi_3 dm\, dn \left(mn(m+n)\frac{\sqrt{3}}{48} \right), \tag{51}$$

where ϕ_i and π_i (the so called "Darboux" variables) are canonically conjugate variables [12]. If the integrand depends only on m and n, we can carry out the integral over the angles $d\phi_i$ and their canonically conjugate momenta $d\pi_i$. One obtains (see appendix B for details of the derivation) [26]

$$\int \prod_{i=1}^{3} d\phi_i\, d\pi_i = \frac{(2\pi)^3}{2}\, mn(m+n)\,. \tag{52}$$

With this result, we can write Eq. (51) as

$$\int d^8Q = \frac{(2\pi)^3}{32\sqrt{3}} \int dm\, dn \left(m^2 n^2 (m+n)^2 \right)\,. \tag{53}$$

Substituting Eq. (53) into Eq. (49), we obtain Eq. (50).

In summary, we have shown in this section that for a sum over classical $SU(3)$ representations, the measure can be expressed as an eight dimensional integral over the components of the classical color charge $\mathbf{Q}$ (with a magnitude of order $\sqrt{k} \gg 1$).

---

[12]It can be checked that $Q_1 \cdots Q_8$ satisfy $\{Q_a, Q_b\}_{\mathrm{PB}} = f_{abc}Q_c$ where the Poisson Bracket is defined as $\{Q_a, Q_b\}_{\mathrm{PB}} = \sum_i^3 \left(\frac{\partial Q_a}{\partial \phi_i} \frac{\partial Q_b}{\partial \pi_i} - \frac{\partial Q_a}{\partial \pi_i} \frac{\partial Q_b}{\partial \phi_i} \right)$.

4.1.1   From random walks to path integrals in $SU(3)$ QCD

The reader may note that Eq. (50) is exactly analogous to Eq. (25) obtained for $SU(2)$ quark sources. Then replacing $\mathbf{l} \to \mathbf{Q}$ in Eqs.(30), (31) and (32), we recover

$$\int \prod_a d\rho^a \, W[\boldsymbol{\rho}] \equiv \int \prod_a d\rho^a \, \exp\left(-\int_A d^2 x_\perp \, \frac{\boldsymbol{\rho}(x_\perp) \cdot \boldsymbol{\rho}(x_\perp)}{2\mu_A^2}\right) \tag{54}$$

where now $a = 1, \cdots, 8$ and

$$\mu_A^2 = \frac{g^2 A}{2\pi R_A^2} \, , \tag{55}$$

is the color charge squared per unit area and is independent of N_c as stated in the previous section. Thus, even in the $SU(3)$ case, and as conjectured in the MV model, the leading contribution to the path integral measure over classical color charge densities has a Gaussian weight proportional to the quadratic Casimir. As shown in Eq. (45), the contribution of the cubic Casimir to the weight is $O(1/\sqrt{k})$ which vanishes when $k \to \infty$.

4.2   Many $SU(3)$ gluons

In the case of uncorrelated gluons, we are multiplying the **8**'s. Since the Young tableaux for one **8** has three boxes, we add three boxes at a time as we build up higher dimensional representations. As a consequence, there are sites in the (m, n) plane we will never visit in the random walk. One can see this as follows. In general, a Young tableaux diagram for a state (m, n) will have $m + 2n + 3p$ boxes, where p is an integer with $p \geq 0$. Therefore, since we are multiplying boxes in multiples of 3, unless $m + 2n = 3l$ (where again, l is an integer ≥ 0), the site (m, n) can never be reached from the origin. In

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general, the allowed states in the (m, n) plane have the form $(3s + u, 3t + u)$, where s, t are integers ≥ 0, and $0 \leq u \leq 2$. Thus for example, the state $(1, 1)$ is allowed but not $(1, 2)$ or $(1, 0)$.
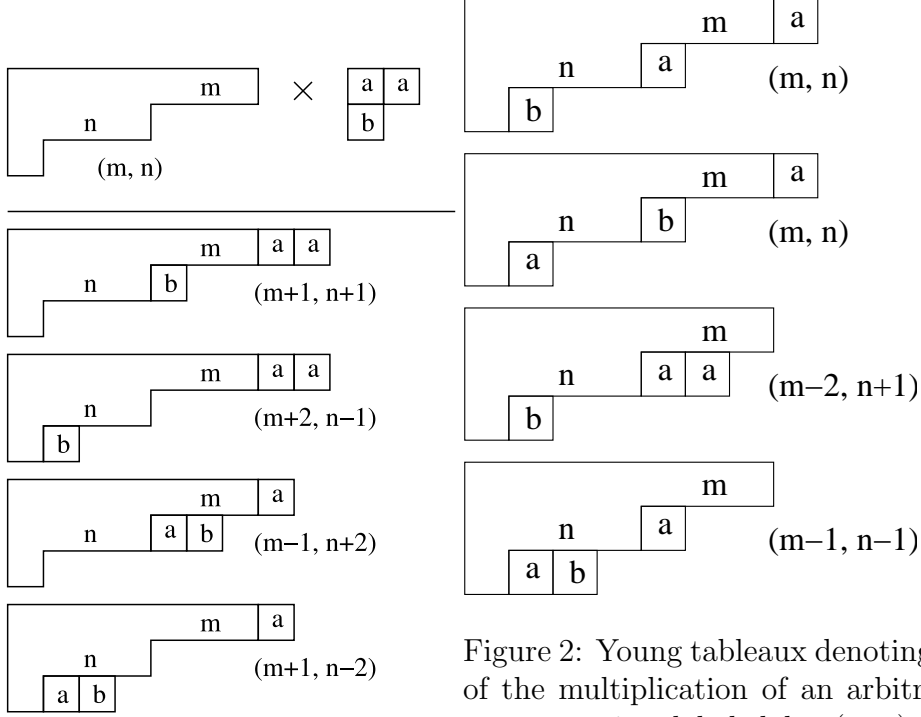

Figure 2: Young tableaux denoting the result of the multiplication of an arbitrary $SU(3)$ representation labeled by (m,n) with an **8** representation

Fig. 2 shows Young tableaux illustrating the representations generated by multiplying an arbitrary $SU(3)$ representation (m, n) with a $(1, 1)$ (or **8**) representation. It is clear from the multiplication of Young tableaux that one obtains

$$(\mathbf{1}, \mathbf{1}) \times (\boldsymbol{m}, \boldsymbol{n})$$

$$= \ (\boldsymbol{m} + \mathbf{1}, \boldsymbol{n} + \mathbf{1}) + (\boldsymbol{m} + \mathbf{2}, \boldsymbol{n} - \mathbf{1})$$

$$+ (\boldsymbol{m} + \mathbf{1}, \boldsymbol{n} - \mathbf{2}) + (\boldsymbol{m} - \mathbf{1}, \boldsymbol{n} + \mathbf{2})$$

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$$+ (m, n) + (m, n) + (m - 2, n + 1) + (m - 1, n - 1) \tag{56}$$

Just as in the case of many quarks we dealt with previously, we will denote the multiplicity of a given state (m, n) by a matrix $N_{m,n}$ with $m, n \geq 0$. Our initial state matrix is

$$N_{0,0}^{(0)} = 1 \quad \text{otherwise} \quad N_{m,n}^{(0)} = 0 \tag{57}$$

From Eq. (56), one can deduce immediately that the multiplicity of states is given by recursion relation,

$$
\begin{aligned}
N_{m,n}^{(k+1)} &= 2\,N_{m,n}^{(k)} + N_{m+2,n-1}^{(k)} + N_{m+1,n+1}^{(k)} \\
&\quad + N_{m+1,n-2}^{(k)} + N_{m-1,n+2}^{(k)} + N_{m-1,n-1}^{(k)} + N_{m-2,n+1}^{(k)}
\end{aligned} \tag{58}
$$

Note that only states with m and $n \geq 0$ are allowed, so in the above, the multiplicity of states whose labels are negative is zero.

There are a few special cases which do not satisfy the above recursion relation. These are the states that lie on the edges of the first quadrant, namely, $(m > 0, 0)$ and $(0, n > 0)$ and at the origin $(0, 0)$. These satisfy the following relations:

$$N_{0,0}^{(k+1)} = N_{1,1}^{(k)}, \tag{59}$$

This is because when repeatedly multiplying the **8**'s, a singlet state can only come from **8** $\times$ **8**. For $m > 0$ and $n > 0$,

$$
\begin{aligned}
N_{m,0}^{(k+1)} &= N_{m,0}^{(k)} + N_{m+1,1}^{(k)} + N_{m-1,2}^{(k)} + N_{m-2,1}^{(k)} \tag{60} \\
N_{0,n}^{(k+1)} &= N_{0,n}^{(k)} + N_{2,n-1}^{(k)} + N_{1,n+1}^{(k)} + N_{1,n-2}^{(k)} \tag{61}
\end{aligned}
$$

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From Eq. (58), one can guess that the basic unit in solving the recursion relation should be

$$C_{k:\, l_1,l_2,\cdots,l_6} \equiv \frac{k!}{l_1!\, l_2!\,\cdots l_6!\,(k-\sum_{i=1}^{6} l_i)!} 2^{k-\sum_{i=1}^{6} l_i}, \qquad (62)$$

which appears in the expansion of $8^k = (1+1+1+1+1+1+2)^k$.

Following the pattern established thus far in Eq. (58), we set

$$(m,n) = l_1(1,1) + l_2(-1,-1) + l_3(2,-1) + l_4(-2,1) + l_5(1,-2) + l_6(-1,2) \qquad (63)$$

One can then define the following function (again in analogy to section 4.1),

$$G_{k:\, m,n} = \sum_{l_1,\cdots,l_6=0}^{k} C_{k:\, l_1,l_2,\cdots,l_6}\, \delta_{m-[(l_1-l_2)+2(l_3-l_4)+(l_5-l_6)]}\, \delta_{n-[(l_1-l_2)-(l_3-l_4)-2(l_5-l_6)]} \qquad (64)$$

The solution to Eqs.(58), (59), and (61) is again given by the same combination of $G_{k:\, m,n}$'s as in the quark case

$$\begin{aligned}
N^{(k)}_{m,n} &= G_{k:\, m,n} + G_{k:\, m+3,n} + G_{k:\, m,n+3} \\
&\quad - G_{k:\, m+2,n-1} - G_{k:\, m-1,n+2} - G_{k:\, m+2,n+2}. \qquad (65)
\end{aligned}$$

The validity of this form of solution has been checked extensively by numerical means. Namely, we generated $N^{(k)}_{m,n}$ numerically by using the recursion relations and compared it with the values obtained by the solution (65). We are therefore confident that Eq. (65) is the solution.

To express $G_{k:\, m,n}$ in terms of the SU(3) Casimirs, we can use the following integral representation of the Kronecker delta:

$$\delta_{s,t} = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\, e^{i\phi(s-t)}, \qquad (66)$$

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which enables us to carry out the sum in Eq. (64). This yields

$$G_{k:\,m,n} = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi}$$
$$\times\, e^{i\phi m + i\varphi n} \left(2 + 2\cos(\phi + \varphi) + 2\cos(2\phi - \varphi) + 2\cos(2\varphi - \phi) \right)^k \quad (67)$$

Using the small angle approximation (valid for large m, n), one obtains,

$$
\begin{aligned}
G_{k:\,m,n} &\approx 8^k \frac{1}{2\pi} \int_{-\infty}^{\infty} d\phi\, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\varphi\, e^{i\phi m + i\varphi n} \left(1 - (3/4)(\phi^2 + \varphi^2 - \phi\varphi) \right)^k \\
&\approx 8^k \frac{1}{2\pi} \int_{-\infty}^{\infty} d\phi\, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\varphi\, e^{i\phi m + i\varphi n}\, e^{-(3k/4)(\phi^2 + \varphi^2 - \phi\varphi)} \\
&= \frac{8^k}{(2\pi)^2} \frac{8\pi}{3\sqrt{3}k} e^{-4(m^2 + n^2 + mn)/(9k)} .
\end{aligned}
\quad (68)
$$

Substituting this expression into Eq. (65) and applying the Taylor expansion technique, identically as in Eq. (46), we obtain for the multiplicity distribution of k gluons

$$N_{m,n}^{(k)} = \frac{8^k}{(2\pi)^2} \frac{512\, m\, n\, (m+n)\, \pi}{81\sqrt{3}\, k^4} \exp\left(-\frac{C_F}{C_A} \frac{N_c}{k} D_2^{m,n} \right) \quad (69)$$

Eq. (69) is the final result of this sub-section. It has several interesting features. Firstly, unlike the case of many quarks, it has no dependence whatsoever on the cubic Casimir. Indeed, the result is entirely symmetric in m and n. Secondly, the weight in the exponential is just simply a Gaussian in the color charges. These are of order $\sqrt{k} \gg 1$, and are therefore classical color charges. The prefactor, as in the $SU(3)$ quark case, is proportional to $m\, n\, (m+n)$. Therefore, as in the $SU(3)$ case, one can write the sum over color charges in a box as an integral over an 8-dimensional classical measure. We further note that for the gluons, the total number of degrees of freedom does turn out to be 8^k even in our large k approximation. Finally,

the variance of the quadratic Casimir is proportional to an additional factor of C_A/C_F compared to the quark case. One thus obtains for the argument of the exponential (see Eq. (32)),

$$
\begin{aligned}
\frac{C_F}{C_A}\frac{C_A}{k}\mathbf{Q}^2 &= \frac{C_F}{C_A}C_A\frac{(\Delta x_\perp)^4}{g^2 k}\rho^a\rho^a \\
&= \frac{1}{2\,N_c}\frac{\pi R^2}{g^2 A}(\Delta x_\perp)^2\rho^a\rho^a\,,
\end{aligned}
\tag{70}
$$

where, in our simple model k is now the number of gluons in a tube of transverse size $(\Delta x_\perp)^2$, namely, $k = (N_c^2 - 1)\,A\,(\Delta x_\perp)^2/\pi R^2$. One can thus deduce that the color charge squared per unit area of the gluon charges is

$$
\mu_{A,\text{glue}}^2 = g^2\,N_c\,\frac{A}{\pi R^2}\,.
\tag{71}
$$

This result for the gluons is precisely the one expected in the MV model [28].

The rest of the derivation of the path integral for the MV-effective action goes through as discussed previously. We end this discussion with a caveat. It has long been understood that a renormalization group treatment where one includes bremsstrahlung gluons from quark sources, as additional sources for further bremsstrahlung of softer "wee" gluons, will modify the simple Gaussian of the MV model [20, 29, 21]. Naively, one can interpret this in terms of adding gluons to the color sources. The reason we have a Gaussian distribution of sources in the problem discussed here is two-fold; firstly, we ignore (quantum) correlations in the sources and secondly, the ground state in our problem is a singlet **1** representation. A closer analogy to the bremsstrahlung scenario would be to have quarks in the ground state and then add **8** representations. This would correspond to different initial conditions for our recursion relations and is outside the scope of this paper.

4.3 Many $q\bar{q}$ pairs

In this case, we are multiplying $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8}$ representations. Then, from the gluon case, it is straightforward to see that

$$
\begin{aligned}
((1,0) &\times (0,1)) \times (m, n) \\
&= (m, n) + (m + 1, n + 1) + (m + 2, n - 1) \\
&\quad + (m + 1, n - 2) + (m - 1, n + 2) \\
&\quad + (m, n) + (m, n) + (m - 2, n + 1) + (m - 1, n - 1)
\end{aligned}
\tag{72}
$$

The recursion relation for the multiplicity distribution $N_{m,n}$ is identical with the gluon case with the exception that one replaces $2N_{m,n} \rightarrow 3N_{m,n}$ in Eq. (56). The special cases are also very similar to the gluon case. The basic unit in the solution of Eq. (72) are the "n-nomial coefficients" in the expansion of the 9^k states as $9^k = (1 + 1 + 1 + 1 + 1 + 1 + 3)^k$.

The rest of the derivation is very similar to the previous cases and we will merely quote the result here. We obtain in the large k limit,

$$
N_{m,n}^{(k)} \approx \frac{9^k}{(2\pi)^2} \frac{27\sqrt{3}\, m\, n\, (m + n)\, \pi}{8\, k^4} e^{-\frac{m^2 + m\, n + n^2}{2\, k}}
\tag{73}
$$

Again, one finds an expression symmetric in m and n, the ubiquitous factor $m\, n\, (m + n)$ in the prefactor, and the Gaussian distribution of color charges. Since $D_2^{m,n} \approx (m^2 + n^2 + mn)/3$, one can now easily deduce that $\mu_{A,q\bar{q}}^2 = 2\,\mu_A^2$.

5 $SU(N_c)$

We can carry out a similar analysis of the multiplicity of higher dimensional representations for the general $SU(N_c)$ case. For problems in the standard

model, it is sufficient to have explicit solutions for $SU(2)$ and $SU(3)$ cases. However, there are several reasons to extend the analysis to the more general N_c case. Firstly, we are presented with an interesting non-trivial random walk problem in $N - 1$ dimensions where the number of possible directions at each step depends on the type of representation being multiplied at that step. Secondly, we would like to confirm that the patterns we see in the $N_c = 2$ and $N_c = 3$ cases are not accidental but follow from a more general solution. For example, we interpreted the factors 2 and 3 that multiply the quadratic Casimirs in Eqs. (26) and (49) as the corresponding N_c factors. We would like to verify in this section that they indeed are N_c factors and not merely an accident of combinatorics. Another pattern we observed in previous sections is that the multiplicity is predominantly determined by the quadratic Casimir and that the most probable distribution has weights of order $O(1/\sqrt{k})$. We also would like to see that this is a general pattern. Finally, the large N_c limit constitutes another classical limit of a spin system. Although this large N_c limit is not what one needs in the small x MV model, working out an explicit expression for a general N_c may shed some light on when exactly we can regard a system as a "classical" system.

5.1 N_c

Let us consider adding a large number of quarks in the fundamental representation of $SU(N_c)$. For $SU(N_c)$, a representation $\boldsymbol{R}$ can be uniquely labeled by an integer vector

$$\boldsymbol{R} = (r_1, r_2, \cdots, r_{N_c-1}) . \tag{74}$$

The corresponding Young tableau has N_c rows with f_i number of boxes in the i-th row. The number of boxes in the rows and r_i are related by $r_i = f_i - f_{i+1}$ and f_i's should satisfy the inequality

$$f_1 \geq f_2 \geq \cdots f_{N_c-1} \geq f_{N_c} \tag{75}$$

since $r_i \geq 0$. In this notation, a single quark is labeled by

$$\boldsymbol{N_c} = (1, 0, \cdots, 0) \tag{76}$$

We begin with a random representation given by

$$\boldsymbol{R} = (r_1, r_2, \cdots, r_{N_c-1}) \tag{77}$$

We would like to see what irreducible representations are generated when $\boldsymbol{N}_c$ and $\boldsymbol{R}$ are multiplied together.

Multiplying by $\boldsymbol{N}_c$ corresponds to adding a box to the Young tableau for $\boldsymbol{R}$. Adding a box to the i-th row lengthens it by 1 or $f_i \to f_i + 1$. Therefore the labels change as

$$r_i = f_i - f_{i+1} \to r_i + 1 \tag{78}$$

$$r_{i-1} = f_{i-1} - f_i \to r_i - 1 \tag{79}$$

The result of the multiplication by N_c is

$$\boldsymbol{R} \times \boldsymbol{N_c} \;=\; \sum_{m=1}^{N_c} \boldsymbol{R}'_m \tag{80}$$

where

$$\boldsymbol{R}'_m \;\equiv\; (r_1, r_2, \cdots, r_{N_c-1}) - \mathbf{e}_{m-1} + \mathbf{e}_m \tag{81}$$

Here $\mathbf{e}_m$ is a unit vector in the m-th direction (with $\mathbf{e}_0 = \mathbf{e}_{N_c} = 0$). As before, $\boldsymbol{R}'_m$ with negative entries should be discarded. If one keeps multiplying $\boldsymbol{N}_c$'s, the multiplicities of the representations in the k-th iteration and $(k-1)$-th iteration are related by

$$G^{(k)}(\boldsymbol{R}) = \sum_{m=1}^{N_c} G^{(k-1)}(\boldsymbol{R} + \mathbf{e}_{m-1} - \mathbf{e}_m) \tag{82}$$

If $k \gg 1$ and $r_i \gg 1$, we can make the continuum approximation by Taylor expanding the right hand side and discarding higher derivative terms:

$$G^{(k)}(\boldsymbol{R}) \approx N_c\, G^{(k-1)}(\boldsymbol{R}) + \hat{D}_{N_c} G^{(k-1)}(\boldsymbol{R}) \tag{83}$$

where we defined

$$\hat{D}_{N_c} \equiv \sum_{m=1}^{N_c-1} \partial_m^2 - \sum_{m=1}^{N_c-2} \partial_m \partial_{m+1}\,, \tag{84}$$

with $\partial_s = \partial/\partial r_s$.

We also would like to make a similar continuum approximation for k as well so that Eq. (83) becomes a partial differential equation. Since N_c can be large, care must be taken to ensure the validity of the Taylor expansion. To do so, we first let

$$G^{(k)}(\boldsymbol{R}) = G(k, \boldsymbol{R}) = N_c^k \sigma(k, \boldsymbol{R})\,, \tag{85}$$

to take care of the potentially large N_c dependence. In the large k limit, the equation for $\sigma(k, \boldsymbol{R})$ is then

$$\partial_k \sigma = \frac{1}{N_c} \hat{D}_{N_c} \sigma \tag{86}$$

Letting the expressions we obtained for the $SU(2)$ and $SU(3)$ cases guide us, we try the following ansatz:

$$\sigma(k, \boldsymbol{R}) = g(k) \exp\left(-\frac{D_2(\boldsymbol{R})}{rk}\right) , \tag{87}$$

where $D_2(\boldsymbol{R})$ is the quadratic Casimir of the representation $\boldsymbol{R}$ and r is a yet to be determined constant. Substituting Eq. (87) into Eq. (86) and solving for $g(k)$ is tedious but straightforward. The solution is

$$G(k, \boldsymbol{R}) = C \frac{N_c^k}{k^{(N_c-1)/2}} \exp\left(-\frac{N_c^2(N_c^2 - 1)}{24k} - N_c \frac{D_2(\boldsymbol{R})}{k}\right) , \tag{88}$$

where C is a normalization constant. Details of this derivation are in Appendix C.

Note the appearance of $N_c D_2/k$ in the exponential. Both the $SU(2)$ and $SU(3)$ cases we worked out match up with this expression and the expression for the color charge squared per unit area obtained in section 3.2

$$\mu_A^2 = \frac{g^2 A}{2\pi R_A^2} , \tag{89}$$

is valid for any N_c. The additional term in the exponential only affects the overall normalization and is negligible when $N_c \ll k$ as in the $SU(2)$ and $SU(3)$ cases.

The expression in Eq. (88) is a particular solution of the recursion relation in the large k limit. However, it is not guaranteed that Eq. (88) satisfies the necessary boundary condition. In fact, the solution (88) corresponds to the $G_{k:m,n}$ functions in the previous section. The multiplicity $N(k, \boldsymbol{R})$ should therefore be a particular linear combination of $G(k, \boldsymbol{R})$ as in Eq. (41).

Unfortunately, we have been unable to find the analog of Eq. (41) for general N_c although it is conceivable that a general solution may be obtained

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along lines presented in Ref.[38] and Ref.[39]. However, if the pattern seen in $SU(2)$ and $SU(3)$ were to persist, one might guess that in the large k limit, the multiplicity should be given by

$$N(k, \boldsymbol{R}) = \mathcal{N}\, d(\boldsymbol{R}) \, \exp\left(-N_c \frac{D_2(\boldsymbol{R})}{k}\right) \tag{90}$$

where $\mathcal{N}$ is a normalization constant and $d(\boldsymbol{R})$ is the dimension of the $\boldsymbol{R}$ representation in $SU(N_c)$. We can also expect the color phase space measure will have the structure

$$d^{N_c^2-1}Q = dr_1 dr_2 \cdots dr_{N_c-1}\, d(\boldsymbol{R})^2 \, d\Omega_{N_c\,(N_c-1)} \tag{91}$$

where $d\Omega_{N_c\,(N_c-1)}$ are the $\boldsymbol{R}$ independent canonically conjugate Darboux variables. We have been unable thus far to verify these conjectures explicitly.

5.2 $\boldsymbol{N}_c \times \bar{\boldsymbol{N}}_c$

We can easily generalize the above analysis to the case of a large number of quark and anti-quark pairs. For $SU(N_c)$, the fundamental and its dual representations are labeled by

$$\boldsymbol{N}_c = (1, 0, \cdots, 0) \tag{92}$$

$$\bar{\boldsymbol{N}}_c = (0, 0, \cdots, 1) \tag{93}$$

As previously, we start with a representation

$$\boldsymbol{R} = (r_1, r_2, \cdots, r_{N_c-1}) \tag{94}$$

Multiplying by $\boldsymbol{N}_c$ results in

$$(r_1', r_2', \cdots, r_{N_c-1}') = (r_1, r_2, \cdots, r_{N_c-1}) - \mathbf{e}_{m-1} + \mathbf{e}_m \tag{95}$$

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for $0 \leq m \leq N_c$. Further multiplying by $\bar{\boldsymbol{N}}_c$ results in

$$(r_1'', r_2'', \cdots, r_{N_c-1}'') = (r_1', r_2', \cdots, r_{N_c-1}') + \mathbf{e}_{n-1} - \mathbf{e}_n \qquad (96)$$

for $0 \leq n \leq N_c$.

Hence the multiplicities in the k-th iteration and the $(k-1)$-th iteration are related by

$$G^{(k)}(\boldsymbol{R}) = \sum_{m,n=1}^{N_c} G^{(k-1)}(\boldsymbol{R} + \mathbf{e}_{m-1} - \mathbf{e}_m - \mathbf{e}_{n-1} + \mathbf{e}_n) \qquad (97)$$

In the continuum limit, where all the r_i are typically large (or k is large),

$$G^{(k)}(\boldsymbol{R}) = N_c^2\, G^{(k-1)}(\boldsymbol{R}) + 2\,N_c\left(\sum_{m=1}^{N_c-1} \partial_m^2 - \sum_{m=1}^{N_c-2} \partial_m \partial_{m+1} \right) G^{(k-1)}(\boldsymbol{R}) \qquad (98)$$

We again let

$$G^{(k-1)}(\boldsymbol{R}) = G(k, \boldsymbol{R}) = N_c^{2k}\, g(k)\, \exp\left(-\frac{D_2(\boldsymbol{R})}{rk} \right) \qquad (99)$$

The solution for the resulting differential equation is

$$G(k, \boldsymbol{R}) = C\, \frac{N_c^{2k}}{k^{(N-1)/2}}\, \exp\left(-\frac{N_c^2(N_c^2 - 1)}{48k} - \frac{N_c\, D_2(\boldsymbol{R})}{2k} \right) \qquad (100)$$

Again, this is a particular solution of the recursion relation. The comments at the end of the last section (after Eq. (88)) are equally applicable here.

5.3 The adjoint representation

The adjoint case isn't much different from the quark-antiquark case. For $SU(N_c)$,

$$(\boldsymbol{N_c^2 - 1}) = (1, 0, \cdots, 0, 1)\,. \qquad (101)$$

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We know that

$$\boldsymbol{N}_c \times \bar{\boldsymbol{N}}_c = \boldsymbol{1} \oplus (\boldsymbol{N_c^2 - 1}) \tag{102}$$

So all we have to do is to change Eq. (97) to

$$G^{(k)}(\boldsymbol{R}) = \sum_{m,n=1}^{N_c} G^{(k-1)}(\boldsymbol{R} + \mathbf{e}_{m-1} - \mathbf{e}_m - \mathbf{e}_{n-1} + \mathbf{e}_n) - G^{(k-1)}(\boldsymbol{R}) \tag{103}$$

and the solution is

$$G(k, \boldsymbol{R}) = \frac{(N_c^2 - 1)^k}{k^{(N_c-1)/2}} \exp\left(-\frac{(N_c^2 - 1)^2}{48k} - \frac{(N_c^2 - 1)D_2(\boldsymbol{R})}{2N_c k}\right) \tag{104}$$

Again, this is a particular solution of the recursion relation. The comments after Eq. (88) also apply here.

6   Discussion and Summary

We have discussed in this paper the distribution of color charge representations generated by k partons in an $SU(N_c)$ gauge theory. Since the partons are random, the problem is a random walk problem in the space spanned by the $SU(N_c)$ Casimirs. We explicitly considered the $SU(2)$ and $SU(3)$ cases before considering the general N_c limit. For all the cases considered, we find that the most likely representation is one of order $O(\sqrt{k})$. The distribution of representations about this representation is given by an exponential in the quadratic Casimir D_2, with a weight proportional to k. In the case of $SU(3)$ quarks, the contribution due to the cubic Casimir is suppressed relative to the leading Gaussian term by $O(1/\sqrt{k})$. Remarkably, for gluons and quark-anti-quark pairs, the result is given entirely in terms of the quadratic Casimir. Our results for $SU(3)$ can be generalized to the $SU(N_c)$ case. Although the

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random walk is now in the space of $N_c - 1$ Casimirs, the distribution of representations is still given by the quadratic Casimir.

Since the most likely representation is of order $O(\sqrt{k})$, one can argue that, in the $\sqrt{k} \to \infty$ limit, the representation is a classical representation. The arguments here are no different from those for the treatment of classical spins. A formal representation of $SU(N_c)$ representations, suitable for path integral formulations, can be made in terms of coherent states [32]. As discussed in appendix A, it has been shown in Ref. [31] that, for large k, one recovers the classical limit.

Our discussion here was formulated in the framework of the McLerran-Venugopalan model for small x physics. The Gaussian distribution of color charge sources and their treatment as classical color charges was assumed in this model. We have provided here a more rigorous basis for these assumptions. The small x behavior of QCD is more complex than the one outlined in the MV model and more sophisticated renormalization group (RG) treatments have been developed [14, 21]. A key feature of the MV model, namely, the lack of correlations among the sources, breaks down in these treatments. Therefore, unsurprisingly in our random walk picture, the distributions are no longer Gaussian. The classical assumption persists however since the most probable representations are likely still classical ones. This assumption deserves further study. Remarkably, in the limit of very small x, a mean field treatment of the RG equations, recovers a Gaussian distribution of sources-albeit a non-local one [12, 13]. Interactions among sources in this limit produces screened charges of size $1/Q_s$, where Q_s is

the saturation scale [13]. In our language, this might mean that the effective charges are random and can therefore be represented by Gaussian sources. It is interesting to speculate whether the combinatorial techniques developed here for higher dimensional representations can be used in numerical studies of high energy Onium-Onium scattering in QCD [15].

Other clear applications of the techniques developed here are in transport problems in QCD. It has been argued previously that the transport of color in high temperature QCD could be represented in terms of the classical dynamics of color charges [27]. It was however not clear when this classical description was the appropriate one and attempts to derive these from first principles have had to resort to ad hoc approximations [36]. We do know though that any systematic treatment of transport problems involves coarse graining of color charges over large distance scales. These can therefore be treated using the recursion techniques developed here. These developments will be discussed elsewhere [37].

Acknowledgments

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[13]In the classical MV model, $Q_s^2 \approx \mu_A^2$, where μ_A^2 is the color charge squared per unit area.

A   Large k color charge representations as classical color charge representations

An interesting limit of quantum theories is the limit of $N \to \infty$, where N can denote either the underlying invariance group of the theory or the size of higher dimensional representations when the invariance group of the theory is held fixed. In both cases, the large N limit corresponds to a classical limit and can be formally shown to correspond to the limit where $\hbar \to 0$. As discussed in a nice review by Yaffe [2], the quantum dynamics of a system will reduce to classical dynamics as $\hbar \to 0$, if and only if the system can be prepared in a state whose uncertainty in its conjugate momenta and positions vanishes in this limit.

Coherent states [30] have this property and it is therefore useful to write states of the quantum theory in this representation. These coherent states are orbits of the Heisenberg-Weyl group, and can therefore be generalized to any Lie group. For an $SU(N_c)$ gauge theory, one can define coherent states carrying $SU(N_c)$ color charges [32].

In Ref. [31], coherent states for $SU(N_c)$ groups were studied with the particular purpose of studying the classical limit. The authors identify co-

herent states of a particular representation in terms of the signature of representation (which is none other than size of the totally symmetric tensor-in our case k [33]) and the weight vectors $\mathbf{n}$ of the representation. They define the variance of the square of the length of the isospin vector (the quadratic Casimir $D_2 = \sum Q_a^2$),

$$\Delta D_2 = \langle \Psi | \sum_a Q_a^2 | \Psi \rangle - \sum_a \langle \Psi | Q_a | \Psi \rangle^2 \equiv \langle \Psi | D_2 - D_2' | \Psi \rangle \,, \qquad (105)$$

In the coherent state basis for fixed $SU(N_c)$,

$$\frac{\Delta D_2}{D_2} = \frac{N_c}{N_c + k} \,. \qquad (106)$$

This result is valid for $k \gg N_c$. Thus the large k limit, where the variance vanishes can be identified with the $\hbar \to 0$ limit in quantum mechanics, and one can formally relate $\hbar = 1/k$. We refer interested readers to Ref. [31] for a more extensive discussion of this correspondence.

B The volume of the canonical phase space in SU(3)

In Refs.[26, 27], the phase space variables for $SU(3)$ are parameterized as

$$\begin{aligned}
Q_1 &= \cos\phi_1 \pi_+ \pi_-, \quad Q_2 = \sin\phi_1 \pi_+ \pi_- \\
Q_3 &= \pi_1 \\
Q_4 &= C_{++}\pi_+ A + C_{+-}\pi_- B, \quad Q_5 = S_{++}\pi_+ A + S_{+-}\pi_- B \\
Q_6 &= C_{-+}\pi_- A - C_{--}\pi_+ B, \quad Q_7 = S_{-+}\pi_- A - S_{--}\pi_+ B \\
Q_8 &= \pi_2
\end{aligned} \qquad (107)$$

with

$$\begin{aligned}
\pi_+ &= \sqrt{\pi_3 + \pi_1}, \quad \pi_- = \sqrt{\pi_3 - \pi_1} \\
C_{\pm\pm} &= \cos\left(\frac{1}{2}(\pm\phi_1 + \sqrt{3}\phi_2 \pm \phi_3)\right), \quad S_{\pm\pm} = \sin\left(\frac{1}{2}(\pm\phi_1 + \sqrt{3}\phi_2 \pm \phi_3)\right),
\end{aligned}$$

(108)

and

$$\begin{aligned}
A &= \frac{1}{2\pi_3}\sqrt{\left(\frac{J_1 - J_2}{3} + \pi_3 + \frac{\pi_2}{\sqrt{3}}\right)\left(\frac{J_1 + 2J_2}{3} + \pi_3 + \frac{\pi_2}{\sqrt{3}}\right)\left(\frac{2J_1 + J_2}{3} - \pi_3 - \frac{\pi_2}{\sqrt{3}}\right)} \\
B &= \frac{1}{2\pi_3}\sqrt{\left(\frac{J_2 - J_1}{3} + \pi_3 - \frac{\pi_2}{\sqrt{3}}\right)\left(\frac{J_1 + 2J_2}{3} - \pi_3 + \frac{\pi_2}{\sqrt{3}}\right)\left(\frac{2J_1 + J_2}{3} + \pi_3 - \frac{\pi_2}{\sqrt{3}}\right)}
\end{aligned}$$

(109)

where (J_1, J_2) equals the (m, n) label of a $SU(3)$ representation. The pairs (π_i, ϕ_i) form the canonical pairs. The total phase space volume given for a fixed (m, n) is given by the integration over these canonical variables:

$$\Omega(m, n) = \int_{m,n} d\phi_1 d\pi_1 d\phi_2 d\pi_2 d\phi_3 d\pi_3 \tag{110}$$

Evaluating the integral over the conjugate momenta is not trivial. It can be inferred from Fig. 1 of Johnson's paper [26] but the explicit derivation, as shown below, is quite involved (if straightforward).

The values of the canonical momenta π_i are restricted to make A and B real. To solve for the ranges of π's, it is convenient to define

$$K_1 \equiv \tfrac{2m+n}{3}, \quad K_2 \equiv \tfrac{2n+m}{3}$$

$$x = \pi_3 + \tfrac{\pi_2}{\sqrt{3}}, \quad y = \pi_3 - \tfrac{\pi_2}{\sqrt{3}} \tag{111}$$

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which also gives

$$dxdy = d(\pi_3 + \pi_2/\sqrt{3})d(\pi_3 - \pi_2/\sqrt{3}) = \frac{2}{\sqrt{3}}d\pi_3 d\pi_2 \tag{112}$$

The factors in A and B are

$$A_1 = K_1 - K_2 + x \tag{113}$$
$$A_2 = K_2 + x \tag{114}$$
$$A_3 = K_1 - x \tag{115}$$
$$B_1 = K_2 - K_1 + y \tag{116}$$
$$B_2 = K_2 - y \tag{117}$$
$$B_3 = K_1 + y \tag{118}$$

Potentially, there are 16 sign combinations for the factors A_i and B_i that make the products $A_1 A_2 A_3$ and $B_1 B_2 B_3$ positive. For instance, to make the argument of A positive, one may require all the factors in A to be positive or require that A_1 and A_2 to be negative and A_3 to be positive.

Fortunately, due to the conditions $x + y > 0$, $K_1 > 0$, $K_2 > 0$ and

$$2K_1 - K_2 = m \geq 0 \tag{119}$$
$$2K_2 - K_1 = n \geq 0 \tag{120}$$

one can easily show that the only consistent sign combination is for all A_i and B_i to be positive.

The condition that all A_i are positive yields the range

$$K_2 - K_1 < x < K_1 \tag{121}$$

The condition that all B_i are positive yields the range

$$K_1 - K_2 < y < K_2 \tag{122}$$

then

$$\int_{K_2-K_1}^{K_1} dx \int_{K_1-K_2}^{K_2} dy \int_{-(x+y)/2}^{(x+y)/2} d\pi_1 = \int_{K_2-K_1}^{K_1} dx \int_{K_1-K_2}^{K_2} dy \, (x+y)$$

$$= \frac{mn(m+n)}{2} \tag{123}$$

Equivalently,

$$\int d\pi_1 d\pi_2 d\pi_3 = \frac{\sqrt{3}}{4} mn(m+n) \tag{124}$$

The integrals over the angles can be evaluated in the following way. We first note that

$$\int d^8Q \delta(Q^2 - 1) = \int dQ \, Q^7 \int d\Omega_7 \delta(Q^2 - 1)$$

$$= \frac{\pi^4}{6}, \tag{125}$$

where we have used the fact that the area of a unit sphere in 8-D is $2\pi^4/3!$ [14]. On the other hand, using Eq.(51) this can be also written as

$$\frac{\pi^4}{6} = \int d^8Q \delta(Q^2 - 1)$$

$$= \int d\phi_1 d\phi_2 d\phi_3 d\pi_1 d\pi_2 d\pi_3 dm dn \, mn(m+n) \frac{\sqrt{3}}{48} \delta\left(\frac{m^2 + mn + n^2}{3} - 1\right) \tag{126}$$

We already know the result of the π_i integration from Eq. (124). Therefore,

$$\frac{\pi^4}{6} = \frac{1}{64} \int d\phi_1 d\phi_2 d\phi_3 dm dn \, m^2 n^2 (m+n)^2 \delta\left(\frac{m^2 + mn + n^2}{3} - 1\right) \tag{127}$$

---

[14]For a nice discussion of the volume and area of $SU(N)$ groups, we refer the readers to [34].

The delta function gives

$$m_1 = \frac{-n + \sqrt{3}\sqrt{4 - n^2}}{2} \tag{128}$$

and the condition $m \geq 0$ becomes $n \leq \sqrt{3}$. Hence

$$
\begin{aligned}
\int dm\, dn\, m^2 n^2 (m + n)^2 \delta\left(\frac{m^2 + mn + n^2}{3} - 1\right) &= \int_0^{\sqrt{3}} dn\, \frac{3(m^2 n^2 (m + n)^2)}{2m + n}\bigg|_{m = m_1} \\
&= \int_0^{\sqrt{3}} dn\, \frac{\sqrt{3}\, n^2 \left(-3 + n^2\right)^2}{\sqrt{4 - n^2}} \\
&= \frac{2\pi}{\sqrt{3}},
\end{aligned} \tag{129}
$$

which results in

$$\int d\phi_1 d\phi_2 d\phi_3 = \frac{2(2\pi)^3}{\sqrt{3}}. \tag{130}$$

The total phase space volume defined in Eq. (110) is therefore determined from Eq. (124) and Eq. (130) to be

$$\Omega(m, n) = \frac{(2\pi)^3}{2} mn(m + n). \tag{131}$$

C   Young tableaux and the quadratic Casimir in $SU(N_c)$

In this section, we establish the relationship between a Young tableau and the quadratic Casimir used in the text to obtain $SU(N_c)$ results.

For $SU(N_c)$, a Young tableau has N_c rows. Suppose a Young tableau has f_j boxes in the j-th row. Then this tableau corresponds to a representation labeled by

$$\boldsymbol{R} = (r_1, r_2, \cdots, r_{N_c - 1}) \tag{132}$$

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where $r_i = f_i - f_{i+1}$. Inverting this relationship yields

$$f_i = \sum_{j=i}^{N_c} r_j \,, \tag{133}$$

with

$$r_{N_c} = \frac{1}{N_c} \left(K - \sum_{j=1}^{N_c-1} j r_j \right) \,, \tag{134}$$

where

$$K = \sum_{i=1}^{N_c} f_i \tag{135}$$

is the total number of boxes in the tableau.

According to Okubo [35], the quadratic Casimir for $SU(N_c)$ is given by

$$D_2(\boldsymbol{R}) = \frac{C_2(\boldsymbol{R})}{2} \tag{136}$$

where

$$C_2(\boldsymbol{R}) = K(1 + N_c - K/N_c) + \sum_{j=1}^{N_c} f_j(f_j - 2j) \tag{137}$$

Since D_2 is a Casimir that depends only on $\boldsymbol{R}$, it must be independent of K even if the expression (137) explicitly contains K. The fact that C_2 is in fact independent of K can be easily checked by taking a derivative with respect to K and using that fact that only r_N depends on K.

In the main text, the equation we need to solve has the form

$$\partial_k \sigma(k, \boldsymbol{R}) = \frac{a}{N_c} \hat{D}_{N_c} \sigma(k, \boldsymbol{R}) \tag{138}$$

where

$$\hat{D}_{N_c} \equiv \sum_{m=1}^{N_c-1} \partial_m^2 - \sum_{m=1}^{N_c-2} \partial_m \partial_{m+1} \tag{139}$$

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Our ansatz is

$$\sigma(k, \boldsymbol{R}) = g(k) \exp\left(-\frac{C_2(\boldsymbol{R})}{rk}\right) \tag{140}$$

The left hand side is then

$$\partial_k \sigma = \partial_k g \, e^{-C_2/rk} + \frac{C_2}{rt^2} g \, e^{-C_2/rk} \tag{141}$$

where we used the fact that C_2 is independent of K or equivalently, k. The right hand side is

$$
\begin{aligned}
\hat{D}_{N_c} g \, e^{-C_2/rk} &= g \hat{D}_{N_c} e^{-C_2/rk} \\
&= \frac{1}{r^2 k^2} \left(\sum_{m=1}^{N_c-1} (\partial_m C_2)^2 - \sum_{m=1}^{N_c-2} (\partial_m C_2)(\partial_{m+1} C_2)\right) g e^{-C_2/rk} \\
&\quad - \frac{1}{rk} \left(\sum_{m=1}^{N_c-1} \partial_m^2 C_2 - \sum_{m=1}^{N_c-2} \partial_m \partial_{m+1} C_2\right) g e^{-C_2/rk} \tag{142}
\end{aligned}
$$

Since the expression Eq. (137) depends on f_j, we need to know

$$
\begin{aligned}
\partial_m f_l &= \frac{\partial}{\partial r_m} (r_l + r_{l+1} + \cdots + r_{N_c}) \\
&= (1 - m/N_c) - \theta(l - m) \tag{143}
\end{aligned}
$$

where $\theta(l - m) = 1$ if $l > m$ and $\theta(l - m) = 0$ if $l \leq m$. Therefore,

$$
\begin{aligned}
\partial_m C_2 &= 2 \sum_{l=1}^{N_c} f_l \partial_m f_l - 2 \sum_{l=1}^{N_c} l \partial_m f_l \\
&= 2 \sum_{l=1}^{N_c} f_l \left((1 - m/N_c) - \theta(l - m)\right) - 2 \sum_{l=1}^{N_c} l \left((1 - m/N_c) - \theta(l - m)\right) \\
&= -2(m/N_c)K + m(N_c - m) + 2 \sum_{l=1}^{m} f_l \tag{144}
\end{aligned}
$$

where we used the fact that $\sum_{l=1}^{N_c} f_l = K$. We note that this formula works even if $m = N_c$ or $m = 0$. In that case $\partial_0 C_2 = 0$, $\partial_{N_c} C_2 = 0$.

Second derivatives are

$$
\begin{aligned}
\partial_m^2 C_2 &= \partial_m 2 \sum_{l=1}^{m} f_l = 2\partial_m (K - \sum_{l=m+1}^{N_c} f_l) \\
&= -2 \sum_{l=m+1}^{N_c} \left(-\frac{m}{N_c} \right) \\
&= 2\frac{m(N_c - m)}{N_c}
\end{aligned} \tag{145}
$$

and

$$
\begin{aligned}
\partial_{m+1}\partial_m C_2 &= -2\partial_{m+1} \sum_{l=m+1}^{N_c} f_l \\
&= 2\frac{m(N_c - m - 1)}{N_c}
\end{aligned} \tag{146}
$$

Using these expressions, it is tedious but straightforward to work out

$$
\sum_{m=1}^{N_c-1} \partial_m^2 C_2 - \sum_{m=1}^{N_c-2} \partial_{m+1}\partial_m C_2 = (N_c - 1) \tag{147}
$$

and

$$
\sum_{m=1}^{N_c-1} (\partial_m C_2)^2 - \sum_{m=1}^{N_c-2} (\partial_{m+1}C_2)(\partial_m C_2) = 2\,C_2 + \frac{N_c(N_c^2 - 1)}{6} \tag{148}
$$

The equation (138) then becomes

$$
\partial_k g + \frac{C_2}{rk^2}g = \frac{a\,g}{N_c}\left[\frac{1}{r^2 k^2}\left(2C_2 + \frac{N_c(N_c^2 - 1)}{6} \right) - \frac{1}{rk}(N_c - 1) \right] \tag{149}
$$

We first let

$$
r = \frac{2a}{N_c} \tag{150}
$$

then solve

$$
\partial_k \ln g = \frac{N_c^2(N_c^2 - 1)}{24ak^2} - \frac{1}{2k}(N_c - 1) \tag{151}
$$

which yields

$$
\sigma(k, \boldsymbol{R}) = \frac{S}{k^{(N_c-1)/2}} \exp\left(-\frac{N_c^2(N_c^2 - 1)}{24ak} - N_c\frac{C_2(\boldsymbol{R})}{2ak} \right) \tag{152}
$$

where S is an arbitrary normalization constant.

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